

VOLUME GROWTH, TEMPEREDNESS AND INTEGRABILITY OF MATRIX COEFFICIENTS ON A REAL SPHERICAL SPACE

FRIEDRICH KNOP

*Department Mathematik, Emmy-Noether-Zentrum
FAU Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany*

BERNHARD KRÖTZ

*Universität Paderborn, Institut für Mathematik
Warburger Straße 100, 33098 Paderborn, Germany*

EITAN SAYAG

*Department of Mathematics, Ben Gurion University of the Negev
P.O.B. 653, Be'er Sheva 84105, Israel*

HENRIK SCHLICHTKRULL

*University of Copenhagen, Department of Mathematics
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark*

ABSTRACT. We apply the local structure theorem from [13] and the polar decomposition of [12] to a real spherical space $Z = G/H$ and control the volume growth on Z . We define the Harish-Chandra Schwartz space on Z . We give a geometric criterion to ensure L^p -integrability of matrix coefficients on Z .

E-mail addresses: `friedrich.knop@fau.de`, `bkroetz@gmx.de`,
`eitan.sayag@gmail.com`, `schlicht@math.ku.dk`

Date: January 6, 2016.

2000 Mathematics Subject Classification. 22F30, 22E46, 53C35, 22E40.

The second author was supported by ERC Advanced Investigators Grant HARG 268105.

1. INTRODUCTION

Consider a real algebraic homogeneous space $Z = G/H$, attached to an algebraic real reductive group G and assumed to carry a G -invariant measure. For a compact symmetric neighborhood B of $\mathbf{1}$ in G we define the volume weight:

$$\mathbf{v}(z) := \text{vol}_Z(Bz) \quad (z \in Z).$$

The choice of the ball B is not important, as different balls yield equivalent weights (mutual ratios are bounded above and below by positive constants). Our first aim is then to study the *volume growth* of Z , the growth of \mathbf{v} as a function of $z \in Z$.

In order to efficiently control the volume growth we need a sufficiently explicit parametrization of Z . This is possible in case Z is *real spherical*, that is, minimal parabolic subgroups $P < G$ admit open orbits on Z . We assume this, and choose P such that its orbit through the standard base point $z_0 := H$ is open.

It was shown in [12] that then there is a split torus $A_Z < P$, a closed simplicial cone $A_Z^- \subset A_Z$ (called the compression cone), a compact set $\Omega \subset G$ and a finite set $F \subset G$ such that Z admits the generalized polar decomposition

$$(1.1) \quad Z = \Omega A_Z^- F \cdot z_0.$$

Since the compact set Ω can be incorporated in B , it is then sufficient to find bounds for the function $a \mapsto \mathbf{v}(Baf \cdot z_0)$ on the compression cone A_Z^- , for each $f \in F$. Applying the local structure theorem from [13] we obtain in Proposition 4.3 the precise asymptotic behavior of \mathbf{v} , namely

$$\mathbf{v}(af \cdot z_0) \asymp a^{-2\rho_{\mathfrak{u}}} \quad (a \in A_Z^-, f \in F).$$

for a suitably defined exponent $\rho_{\mathfrak{u}}$ on \mathfrak{a}_Z .

The polar decomposition (1.1) leads to a notion of *temperedness* for unimodular real spherical spaces. This is accomplished with a function space $\mathcal{C}(Z)$ defined as follows. Using (1.1) another weight function on Z is defined by

$$\mathbf{w}(z) := \sup \|\log a\|, \quad (z \in Z),$$

where the supremum is taken over all $a \in A_Z^-$ such that $z \in \Omega a F \cdot z_0$. We then consider the family of semi-norms

$$p_n(f) := \left(\int_Z |f(z)|^2 (1 + \mathbf{w}(z))^n dz \right)^{\frac{1}{2}} \quad (n \in \mathbb{N})$$

on $C_c(Z)$, with which we define a Fréchet completion E . Here G acts by the regular representation. Now $\mathcal{C}(Z)$ is defined as the set E^∞ of

smooth vectors. We refer to it as the *Harish-Chandra Schwartz space* on Z , thus generalizing a notion for symmetric spaces from [1].

In Proposition 5.1 we give a different characterization of $\mathcal{C}(Z)$ by means of the volume weight \mathbf{v} . For $f \in C_c(Z)$ let

$$q_n(f) := \sup_{z \in Z} |f(z)| \sqrt{\mathbf{v}(z)} (1 + \mathbf{w}(z))^n \quad (n \in \mathbb{N}).$$

Using a result from [2] we show that $\mathcal{C}(Z)$ coincides with the space of smooth vectors for the Fréchet completion obtained from this family of seminorms. The general results of [2] imply further that $\mathcal{C}(Z)$ is nuclear.

Finally we investigate L^p -integrability of generalized matrix coefficients. Here we are given a unitary representation (π, \mathcal{H}) of G and an H -invariant distribution vector $\eta \in \mathcal{H}^{-\infty} := (\mathcal{H}^\infty)'$. Every smooth vector $v \in \mathcal{H}^\infty$ then gives rise to a smooth function on Z , the generalized matrix-coefficient associated to v and η :

$$m_{v,\eta}(g \cdot z_0) = \eta(\pi(g^{-1})v) \quad (g \in G).$$

We denote by $H_\eta \supset H$ the full stabilizer of η , and note that the matrix coefficient factorizes to a function on G/H_η , which we denote again by $m_{v,\eta}$. It can happen that H_η is strictly larger than H : take for example π the trivial representation. Less pathological examples occur for the triple spaces $G/H = G_0 \times G_0 \times G_0 / \text{diag}(G_0)$ with G_0 a Lorentzian group. These are real spherical, and it happens for some pairs (π, η) that $H \subsetneq H_\eta \subsetneq G$.

We say that Z has *property (I)* provided that for all unitary irreducible representations (π, \mathcal{H}) and all $\eta \in (\mathcal{H}^{-\infty})^H$ the following holds:

- (1) $Z_\eta := G/H_\eta$ carries a G -invariant measure.
- (2) There exists $1 \leq p < \infty$ such that for all $v \in \mathcal{H}^\infty$ the matrix coefficient $m_{v,\eta}$ belongs to $L^p(Z_\eta)$.

According to [12] there always exists a maximal split torus A in P such that $A_Z \subset A$ and

$$(1.2) \quad A_Z^- \cdot z_0 \supset A^- \cdot z_0,$$

with $A^- \subset A$ the closure of the negative chamber determined by the unipotent radical N of P . If this is accomplished with equality in (1.2), then Z is called *wavefront*. Symmetric spaces, for instance, are wavefront. The main result on L^p -integrability is then:

Theorem 1.1. *Let $Z = G/H$ be a wavefront real spherical space with H self-normalizing and reductive. Then Z has property (I).*

This result will be applied to a lattice counting problem in a subsequent publication [17].

2. HOMOGENEOUS REAL SPHERICAL SPACES

Real Lie groups in this paper will be denoted by upper case Latin letters, A, B etc, and their associated Lie algebras with the corresponding lower case Gothic letter $\mathfrak{a}, \mathfrak{b}$ etc. The identity component of a real Lie group A is denoted by A_0 .

Throughout we assume that G is an algebraic real reductive group, i.e. there is a connected complex reductive group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $G \subset G_{\mathbb{C}}$.

Let $H < G$ be a subgroup with finitely many components and form the homogeneous space $Z = G/H$. We assume that Z is *real algebraic*, i.e. there is complex algebraic subgroup $H_{\mathbb{C}} < G_{\mathbb{C}}$ such that $G \cap H_{\mathbb{C}} = H$. Note that $Z_{\mathbb{C}} := G_{\mathbb{C}}/H_{\mathbb{C}}$ carries naturally the structure of a complex $G_{\mathbb{C}}$ -variety for which we have a G -equivariant embedding

$$Z \hookrightarrow Z_{\mathbb{C}}, \quad gH \mapsto gH_{\mathbb{C}}.$$

Observe that Z is a union of connected components (with respect to the Euclidean topology) of the real points $Z_{\mathbb{C}}(\mathbb{R})$ of $Z_{\mathbb{C}}$.

With the letter P we denote a minimal parabolic subgroup $P < G$. We call Z *real spherical* provided that P admits an open orbit on Z . We assume this, and that P is chosen so that PH is open in G .

For a reductive Lie algebra \mathfrak{g} we write \mathfrak{g}_n for the direct sum of the non-compact ideals in $[\mathfrak{g}, \mathfrak{g}]$. If \mathfrak{g} is the Lie algebra of G , then G_n denotes the corresponding connected normal subgroup of $[G, G]$.

We recall the local structure theorem of [13]: There exists a unique parabolic subgroup $Q \supset P$ with Levi-decomposition $Q = LU$ such that:

- $QH = PH$.
- $L_n \subset Q \cap H = L \cap H$.
- The map

$$(2.1) \quad Q \times_L (L/L \cap H) \rightarrow P \cdot z_0, \quad [q, l(L \cap H)] \mapsto qlH$$

is a Q -equivariant diffeomorphism.

Note that $D := L/L_n$ is a Lie group with compact Lie algebra \mathfrak{d} . Observe that $\mathfrak{d} := \mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_c$ where $\mathfrak{z}(\mathfrak{l})$ is the center of \mathfrak{l} and \mathfrak{l}_c the direct sum of all compact simple ideals in \mathfrak{l} . Hence $L/L \cap H$ is a homogeneous space for the group D , say D/C , and it follows from (2.1) that the map

$$(2.2) \quad U \times D/C \rightarrow P \cdot z_0$$

is a diffeomorphism. As D is algebraic it decomposes as a product of subgroups

$$(2.3) \quad D = A_D \times M_D$$

with $A_D \simeq (\mathbb{R}^+)^n$ a connected real torus and M_D a compact group (observe that A_D is the connected component of an algebraic torus). As C is an algebraic subgroup of D we have $C = A_C \times M_C$ with $A_C = A_D \cap C$ and $M_C = M_D \cap C$. We set $A_Z := A_D/A_C$ and $M_Z = M_D/M_C$ and record

$$D/C \simeq A_Z \times M_Z$$

Then $\dim \mathfrak{a}_Z$ is an invariant of Z which we call its *real rank*, see [13].

Let $L = K_L A_L N_L$ be an Iwasawa decomposition of L , and let $G = KAN$ be an Iwasawa decomposition of G which is compatible, that is, $K_L = K \cap L$, $A = A_L$ and $N_L = N \cap L$. We denote by M the centralizer of \mathfrak{a} in K , and by θ an algebraic Cartan involution of G for which K is the set of θ -fixed points.

Write $\Sigma^+ \subset \mathfrak{a}^*$ for the set of positive roots attached to \mathfrak{n} and set

$$\mathfrak{a}^- = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma^+) \alpha(X) \leq 0\}$$

for the closure of the negative Weyl chamber.

Set $A_H := A \cap H$, $M_H = M \cap H$ and note that $A_Z \simeq A/A_H$. Note that $A_Z = \exp(\mathfrak{a}_Z)$ for $\mathfrak{a}_Z = \mathfrak{a}/\mathfrak{a} \cap \mathfrak{h}$, and that A_H is connected as A has no torsion elements. In the introduction we realized A_Z as a subgroup of A with Lie algebra $\mathfrak{a}_Z \subset \mathfrak{a}$. As that requires the choice of a complement, the realization as a quotient is preferable.

2.1. The compression cone and the polar decomposition. Let $Z = G/H$ be a homogeneous real spherical space. We recall here a few results of [12], Sect 5, about the compression (or valuation) cone $\mathfrak{a}_Z^- \subset \mathfrak{a}_Z$ of Z .

Note that $P_{\mathbb{C}}H_{\mathbb{C}} \subset G_{\mathbb{C}}$ is a Zariski open which is affine by the local structure theorem. We denote by \mathcal{P}_{++} the monoid of regular functions on $G_{\mathbb{C}}$, which are real valued on G , and whose zero set is $G_{\mathbb{C}} \setminus P_{\mathbb{C}}H_{\mathbb{C}}$. Attached to $f \in \mathcal{P}_{++}$ are algebraic characters ψ_f, χ_f of $(H_{\mathbb{C}})_0$ and $P_{\mathbb{C}}$ such that

$$f(ph) = \chi_f(p)\psi_f(h)f(\mathbf{1}) \quad (h \in (H_{\mathbb{C}})_0, p \in P_{\mathbb{C}}).$$

Every $f \in \mathcal{P}_{++}$ gives rise to a finite dimensional irreducible real representation $V_f := \text{span}_{\mathbb{R}}\{R(G)f\} \subset \mathbb{R}[G]$ with $R(g)f := f(\cdot g)$. We write $v_H := f \in V_f$ and note that $[v_H] \in \mathbb{P}(V_f)$ can be chosen such that it is H -fixed (see [12], Lemma 3.11(b)). Let V_f^* be the dual of V_f and let $v_0^* \in V_f^*$ be the evaluation $h \mapsto h(\mathbf{1})$, then $[v_0^*] \in \mathbb{P}(V_f^*)$ is P -fixed, hence a highest weight vector. Let $v_0 \in V_f$ be a lowest weight vector, normalized such that $v_0^*(v_0) = 1$. Assuming also $f(\mathbf{1}) = 1$ we obtain

$$(2.4) \quad f(g) = v_0^*(g \cdot v_H) \quad (g \in G)$$

and in particular for $a \in A$

$$a \cdot v_0 = \chi_f(a)v_0, \quad a \cdot v_0^* = \chi_f(a)^{-1}v_0^*.$$

Hence $\lambda_f := d\chi_f(\mathbf{1}) \in \mathfrak{a}^*$ is the lowest weight of V_f .

Define

$$\mathfrak{a}_f^- := \{X \in \mathfrak{a}_Z \mid \lim_{t \rightarrow \infty} [\exp(tX)v_H] = [v_0]\}$$

and note that this is well defined as $[v_H]$ is H -fixed. It follows from [12] that \mathfrak{a}_f^- is independent of the choice of $f \in \mathcal{P}_{++}$, hence we denote it \mathfrak{a}_Z^- . The closure of \mathfrak{a}_Z^- is denoted by \mathfrak{a}_Z^- and we refer to it as the *compression cone* of Z . Set $A_Z^- := \exp(\mathfrak{a}_Z^-) < A_{\mathbb{C}}/A_{\mathbb{C}} \cap H_{\mathbb{C}}$.

Remark 2.1. (See [12], Sect. 5) The cone \mathfrak{a}_Z^- is finitely generated. For $f \in \mathcal{P}_{++}$ note that $[v_H] \in \mathbb{P}(V_f)$ was chosen to be an H -fixed element. Expand

$$v_H := \sum_{\mu \in \mathfrak{a}^*} v_{\mu}$$

into \mathfrak{a} -weight spaces. Let $\Lambda_f \subset \mathfrak{a}^*$ be the set of $\nu \in \mathfrak{a}^*$ for which $v_{\lambda_f + \nu} \neq 0$. Note that Λ_f is naturally a subset of $\mathfrak{a}_Z^* = \mathfrak{a}_H^{\perp} \subset \mathfrak{a}^*$. Then for $X \in \mathfrak{a}_Z$ it is immediate from the definition that

$$(2.5) \quad X \in \mathfrak{a}_f^- \iff \nu(X) < 0 \quad \text{for all } \nu \in \Lambda_f \setminus \{0\}.$$

Write $\Sigma_{\mathfrak{u}} \subset \Sigma^+$ for the roots with root spaces in \mathfrak{u} . For a root $\alpha \in \Sigma$ we denote by $\alpha^{\vee} \in \mathfrak{a}$ the corresponding co-root. Observe that $\Lambda_f \subset \mathbb{N}_0[\Sigma_{\mathfrak{u}}]$ and that $\lambda_f(\alpha^{\vee}) < 0$ for all $\alpha \in \Sigma_{\mathfrak{u}}$ for $f \in \mathcal{P}_{++}$.

In this paper we use a refinement of the polar decomposition of Z obtained in [12]. We recall from [12] the group

$$J := \{g \in G \mid P_{\mathbb{C}}H_{\mathbb{C}}g = P_{\mathbb{C}}H_{\mathbb{C}}\}$$

which has the properties that $N_G(H) < J$ and $J/N_G(H)$ is compact. Further it was shown that there exists an irreducible finite dimensional representation (π, V) of G with J -fixed vector v_J such that

$$Z_J := G/J \rightarrow \mathbb{P}(V), \quad gJ \mapsto [\pi(g)v_J]$$

defines an embedding. The closure \overline{Z}_J of $Z_J = G/J$ in $\mathbb{P}(V)$ was referred to as simple compactification of Z_J in [12]. The decisive property was that \overline{Z}_J featured a unique closed G -orbit. Passing from Z_J to Z is technically a bit cumbersome as the component group of J is not explicitly known. Instead of using J we prefer here to use a more manageable subgroup $H_1 < G$ which features almost the same properties as J . In order to define H_1 let $\mathfrak{a}_{Z,E} = \mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-)$ be the edge of \mathfrak{a}_Z^- which we realize as a subspace of \mathfrak{a} via the Cartan-Killing form, i.e. $\mathfrak{a}_Z \simeq \mathfrak{a}_H^{\perp} \subset \mathfrak{a}$. Note that $\mathfrak{a}_{Z,E}$ normalizes \mathfrak{h} and therefore

$\mathfrak{h}_1 := \mathfrak{h} + \mathfrak{a}_{Z,E}$ defines a subalgebra. We let $H_{1,\mathbb{C}}$ be the connected algebraic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{h}_{1,\mathbb{C}}$ and set $H_1 := G \cap H_{1,\mathbb{C}}$. Observe that $H_1 < N_G(H)$ and $N_G(H)/H_1$ is compact. Moreover with $A_{Z,E} = \exp(\mathfrak{a}_{Z,E})$ we note that $H_1/HA_{Z,E}$ is a finite group. Let $F_0 \subset H_1$ be a set of representatives of $H_1/HA_{Z,E}$.

Set $Z_1 := G/H_1$ and observe that $\mathfrak{a}_{Z_1} = \mathfrak{a}_Z/\mathfrak{a}_{Z,E}$ and $\mathfrak{a}_{Z_1}^- = \mathfrak{a}_Z^-/\mathfrak{a}_{Z,E}$. Then, according to [11] Sect. 7, there exists a finite dimensional (not necessarily irreducible) representation $\pi : G \rightarrow V$ with H_1 -spherical vector v_{H_1} such that $Z_1 \rightarrow \mathbb{P}(V)$, $gH_1 \mapsto [\pi(g)v_{H_1}]$ embeds into projective space. Moreover, the closure \overline{Z}_1 has a unique closed G -orbit. This is all what is needed to derive in analogy to [12], Section 5, the *polar decomposition* of Z_1 :

$$(2.6) \quad Z_1 = \Omega A_{Z_1}^- F_1 \cdot z_{0,1}$$

where $\Omega \subset G$ is a compact set of the form

$$(2.7) \quad \Omega = F'K$$

for a finite set F' , and $F_1 \subset G$ is a finite set such that

$$(2.8) \quad F_1 \cdot z_{0,1} \subset \exp(i\mathfrak{a}_{Z_1}) \cdot z_{0,1} \subset Z_{1,\mathbb{C}}.$$

Let us mention that F_1 is a set of representatives for the open P -orbits on Z_1 .

It is now a simple matter to lift the polar decomposition from Z_1 to Z (see [14], Section 3.4 and in particular Lemma 3.4). We obtain

$$(2.9) \quad Z = \Omega A_Z^- F \cdot z_0$$

where $\Omega \subset G$ is as above and $F \subset G$ is a finite set such that

$$(2.10) \quad F \cdot z_0 \subset T_Z F_0 \cdot z_0 \subset Z_{\mathbb{C}}$$

with $T_Z := \exp(i\mathfrak{a}_H^\perp) < \exp(i\mathfrak{a})$.

Note that (2.10) implies that $Px \cdot z_0$ is open in Z for each $x \in F$. Indeed, if $g \in T_Z N_G(H)$ then

$$(2.11) \quad \mathfrak{p}_{\mathbb{C}} + \text{Ad}(g)\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}.$$

2.2. The limiting subalgebra. In order to discuss volume growth we need another property of the compression cone. Let $\bar{\mathfrak{u}} = \theta(\mathfrak{u})$ and define

$$\mathfrak{h}_{\text{lim}} = \bar{\mathfrak{u}} + \mathfrak{h} \cap \mathfrak{l},$$

then $\mathfrak{h}_{\text{lim}}$ is a spherical subalgebra with $d = \dim \mathfrak{h} = \dim \mathfrak{h}_{\text{lim}}$. This follows from the decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_Z \oplus \mathfrak{m}_Z \oplus \mathfrak{u} = \bar{\mathfrak{u}} \oplus (\mathfrak{h} \cap \mathfrak{l}) \oplus \mathfrak{a}_z \oplus \mathfrak{m}_Z \oplus \mathfrak{u}.$$

By the same decompositions we find a linear map $T : \bar{\mathfrak{u}} \rightarrow \mathfrak{p}$ such that

$$(2.12) \quad W + Y \mapsto W + Y + T(Y), \quad W \in \mathfrak{l} \cap \mathfrak{h}, Y \in \bar{\mathfrak{u}}$$

provides a linear isomorphism $\mathfrak{h}_{\lim} \rightarrow \mathfrak{h}$.

Remark 2.2. Let $d = \dim \mathfrak{h}$. We write $\text{Gr}_d(\mathfrak{g})$ for the Grassmannian of d -dimensional subspaces of the real vector space \mathfrak{g} and recall from [12], Sect. 5, the following characterization of the compression cone.

Let $X \in \mathfrak{a}_Z$. The following are equivalent:

- (1) $X \in \mathfrak{a}_Z^-$.
- (2) $\lim_{t \rightarrow \infty} e^{t \text{ad } X} \mathfrak{h} = \mathfrak{h}_{\lim}$ in $\text{Gr}_d(\mathfrak{g})$.

2.3. Quasiaffine real spherical spaces. A real spherical space is called quasi-affine if there is a non-zero G -equivariant rational map $Z \rightarrow V$ for a rational real G -module V . If we denote by $\mathcal{P}_{++,\mathbf{1}}$ the subset of \mathcal{P}_{++} which corresponds to right $H_{\mathbb{C}}$ -invariant functions, then quasi-affine means that $\mathcal{P}_{++,\mathbf{1}} \neq \emptyset$.

Note that it is not such a severe restriction to assume that Z is quasi-affine: for $f \in \mathcal{P}_{++}$ with characters (χ_f, ψ_f) one obtains a quasiaffine space $Z_1 = G_1/H_1$ with $G_1 = G \times \mathbb{R}^\times$ and $H_1 := \{(h, \psi_f^{-1}(h)) \mid h \in H\}$. Observe that Z_1 is a G -space and that there is a natural G -fibering

$$\mathbb{R}^\times \rightarrow Z_1 \rightarrow Z.$$

Furthermore we have the following relation between compression cones:

$$(2.13) \quad \mathfrak{a}_{Z_1}^- = \mathfrak{a}_Z^- \oplus \mathbb{R}.$$

3. WEIGHT FUNCTIONS

Recall the notion of weight function from [2]: A positive function w on Z is called a *weight* provided for all compact sets B there exists a constant $C_B \geq 1$ such that

$$w(g \cdot z) \leq C_B w(z) \quad (z \in Z, g \in B).$$

Two positive functions w_1, w_2 on Z are called *equivalent* if the quotient w_1/w_2 is bounded from above and below by positive constants.

For later reference, we note that if w is a weight then so is $\frac{1}{w}$, and likewise the function

$$(3.1) \quad \tilde{w} := \max\{\log w, c - \log w\},$$

when $c > 0$ is a constant.

When Z is quasi-affine one can construct weight functions on it from finite dimensional representations.

Lemma 3.1. *Let Z be quasi-affine, and let (π, V) be a finite dimensional irreducible representation with a non-zero H -fixed vector v_H .*

- (1) The function $w(z) = \|g \cdot v_H\|$ for $z = g \cdot z_0 \in Z$ is a weight.
(2) Assume $V = V_f$ for $f \in \mathcal{P}_{+,1}$ and let $w_f = w$ as in (1). Let $B \subset G$ be compact. Then there exist $C_1, C_2 > 0$ such that

$$C_1 \chi_f(a) \leq w_f(bax \cdot z_0) \leq C_2 \chi_f(a)$$

for all $b \in B$, $a \in A_Z^-$, and $x \in F$.

Proof. (1) is just because π is bounded. For (2) we first observe that (1) implies

$$C_1 w(a) = C_1 w(b^{-1}ba) \leq w(ba) \leq C_2 w(a).$$

Hence we may assume $B = \{1\}$. Next we observe that by Remark 2.1

$$a \cdot v_H = \chi_f(a) \left(v_{\lambda_f} + \sum_{\nu \in \Lambda_f \setminus \{0\}} a^\nu v_{\lambda_f + \nu} \right),$$

which is an orthogonal sum.

Finally we recall from (2.10) that elements x of the finite set F can be written as $x = x_1 x_2$ with $x_1 = t_1 h_1$ with $t_1 \in T_Z$, $h_1 \in H_{\mathbb{C}}$, and $x_2 \in N_G(H)$. Observe that $h_1 x_2 \cdot v_H = c_0 v_H$ for a constant c_0 . Hence

$$ax \cdot v_H = c_0 \chi_f(at_1) \left(v_{\lambda_f} + \sum_{\nu \in \Lambda_f \setminus \{0\}} (at_1)^\nu v_{\lambda_f + \nu} \right),$$

from which it follows that

$$|c_0| \chi_f(a) \|v_{\lambda_f}\| \leq \|ax \cdot v_H\| \leq |c_0| \chi_f(a) \|v_H\|$$

for $a \in A_Z^-$. Here the second inequality is obtained with (2.5). \square

Proposition 3.2. *Let $B_1, B_2 \subset G$ be compact sets. There exists a compact set $B_A \subset A_Z$ such that $a_1 a_2^{-1} \in B_A$ for all pairs $a_1, a_2 \in A_Z^-$ with $B_1 a_1 x_1 \cdot z_0 \cap B_2 a_2 x_2 \cdot z_0 \neq \emptyset$ for some $x_1, x_2 \in F$.*

Proof. It follows from Lemma 3.1 that for each $f \in \mathcal{P}_{+,1}$ there exist $C_1, C_2 > 0$ such that

$$C_1 \chi_f(a_i) \leq w(z) \leq C_2 \chi_f(a_i), \quad i = 1, 2,$$

for all $z \in B_1 a_1 x_1 \cdot z_0 \cap B_2 a_2 x_2 \cdot z_0$. Hence if this set is non-empty then $\chi_f(a_1 a_2^{-1})$ belongs to a compact neighborhood of 1.

We recall from [13] Remark 3.5, that the semigroup generated by the λ_f has rank $\dim \mathfrak{a}_Z$. The proposition follows. \square

We are mainly interested in the compact sets $B \subset G$ which satisfy the polar decomposition

$$(3.2) \quad Z = B A_Z^- F \cdot z_0.$$

and recall from (2.9) that this is the case when $B = \Omega$. By a *ball* in a Lie group we understand a compact symmetric neighborhood of the neutral element. The following is then an immediate consequence of Proposition 3.2.

Corollary 3.3. *For every compact set $B \subset G$ there exists a ball B_A in A_Z such that $BaF \cdot z_0 \subset \Omega(A_Z^- \cap B_A a)F \cdot z_0$ for all $a \in A_Z^-$.*

We fix a norm $\|\cdot\|$ on \mathfrak{a}_Z .

Proposition 3.4. *Let $B \subset G$ be a compact set for which the polar decomposition (3.2) is valid and define*

$$\mathbf{w}(z) = \mathbf{w}_B(z) := \sup_{\{a \in A_Z^- \mid z \in BaF \cdot z_0\}} \|\log a\|, \quad (z \in Z).$$

- (1) *The function $W_B(z) = e^{\mathbf{w}_B(z)}$ is a weight on Z , and all weights obtained in this fashion are equivalent.*
- (2) *There exists a constant $C \in \mathbb{R}$ such that*

$$\|\log a\| \leq \mathbf{w}(\omega a x \cdot z_0) \leq C + \|\log a\|$$

for all $\omega \in B$, $a \in A_Z^-$, $x \in F$.

- (3) *If B is sufficiently large then \mathbf{w} is a weight.*

Proof. (1) Note that (3.2) ensures that the supremum is taken over a non-empty subset of A_Z^- . Moreover, as this set is compact, the supremum is finite.

With (2.13) the proof is easily reduced to the case where Z is quasi-affine. It then follows from Proposition 3.2 that if B_1, B_2 are compact sets both satisfying (3.2) then there exists $C > 0$ such that

$$(3.3) \quad W_{B_2}(z) \leq C W_{B_1}(z)$$

for all $z \in Z$.

It easily follows from (3.3) and the identity

$$W_B(g \cdot z) = W_{g^{-1}B}(z), \quad g \in G, z \in Z.$$

that W_B is a weight. The equivalence also follows from (3.3).

(2) The first inequality is clear. For the second we apply Proposition 3.2 with $B_1 = B_2 = B$. Let $z = \omega a x \cdot z_0$ and let $\omega' \in B$, $a' \in A_Z^-$ and $x' \in F$ be such that $z = \omega' a' x' \cdot z_0$ and $\mathbf{w}(z) = \|\log a'\|$. Then $a' a^{-1} \in B_A$ and hence $\|\log a'\| \leq C + \|\log a\|$ for some $C > 0$ depending only on the size of B_A .

(3) If W is a weight on Z and $\inf_{z \in Z} W(z) > 1$, then $\log W$ is also a weight (see (3.1)). Hence we only need to prove $\inf_{z \in Z} \mathbf{w}_B(z) > 0$. We assume that B contains ΩB_A where B_A is a ball in A . We obtain

for each $z \in Z$ that $z = kax \cdot z_0$ for some $k \in \Omega$, $a \in A_Z^-$, and $x \in F$. Hence it follows from the identity $z = kb^{-1}bax \cdot z_0$ that

$$\mathbf{w}(z) \geq \sup_{\{b \in B_A | ba \in A_Z^-\}} \|\log(ba)\|.$$

This implies the uniform positive lower bound for \mathbf{w} . \square

4. VOLUME GROWTH

In this section and the following we assume that Z is a unimodular real spherical space. We are interested in effective upper and lower bounds of

$$\mathbf{v}_B(g) = \text{vol}_Z(Bg \cdot z_0)$$

in dependence of $g \in G$, for a fixed ball B .

Lemma 4.1. *The function \mathbf{v}_B is a weight and its equivalence class is independent of the ball B .*

Proof. See [2], Lemma-Definition 3.3 and its proof. \square

In the sequel we drop the index B and write \mathbf{v} instead of \mathbf{v}_B . Effective bounds for \mathbf{v} allow a characterization of the tempered spectrum of $L^2(Z)$ as we know from [2] and explicate in the next section when we define the Harish-Chandra Schwartz space $\mathcal{C}(Z)$ of Z .

We shall give lower and upper bounds on $\text{vol}_Z(Ba \cdot z_0)$ for a in A_Z^- . As before we let $Q \supset P$ so that there is an isomorphism (2.2)

$$U \times M_Z \times A_Z \simeq P \cdot z_0.$$

In the sequel we view $M_Z \subset P \cdot z_0$. We define $\rho_{\mathfrak{u}} \in \mathfrak{a}^*$ by $\rho_{\mathfrak{u}}(X) = \frac{1}{2} \text{tr}_{\mathfrak{u}} \text{ad } X$ and note that

Lemma 4.2. $\rho_{\mathfrak{u}} = 0$ on \mathfrak{a}_H .

Proof. Follows, since $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{l}/\mathfrak{a} \cap \mathfrak{h}$ are both unimodular, and $\mathfrak{g}/\mathfrak{h}$ is equivalent to $\mathfrak{u} + \mathfrak{l}/\mathfrak{l} \cap \mathfrak{h}$ as an \mathfrak{a}_H -module. \square

Denote by μ_Z a Haar-measure on Z . It follows that for a continuous function f on $Z = G/H$, compactly supported in $P \cdot z_0$,

$$(4.1) \quad \int_Z f(z) d\mu_Z(z) = \int_U \int_{M_Z} \int_{A_Z} f(uma \cdot z_0) a^{-2\rho_{\mathfrak{u}}} da \, dm \, du$$

where du , dm and da are suitably normalized Haar measures on U , M_Z and A_Z respectively. The integrand makes sense for $a \in A_Z$ because of Lemma 4.2.

As the choice of the ball B does not matter, we may assume that it is invariant under multiplication by $L \cap K$ from the right (for example, we

can replace B by KBK). It then follows, see (2.3), that $BaM_Z = Ba \cdot z_0$ for all $a \in A_Z$. Then

$$(B \cap U)(B \cap A_Z)aM_Z \subset B^2a \cdot z_0$$

which, in view of (4.1) yields a constant $C > 0$ such that

$$C \cdot a^{-2\rho_u} \leq \text{vol}_Z(Ba \cdot z_0) \quad (a \in A_Z).$$

An upper bound is obtained on the negative chamber as follows.

Proposition 4.3. *For every ball B in G there exist constants $C_1, C_2 > 0$ such that*

$$(4.2) \quad C_1 \cdot a^{-2\rho_u} \leq \mathbf{v}(ax \cdot z_0) \leq C_2 \cdot a^{-2\rho_u} \quad (a \in A_Z^-, x \in F)$$

Before giving the proof we note the following result.

Lemma 4.4. *Let $U \subset G$ and $D \subset H$ be open sets. Then*

$$\text{vol}_Z(U \cdot z_0) \text{vol}_H(D) \leq \text{vol}_G(UD).$$

Proof. We exhaust $U \cdot z_0$ by a countable union of disjoint sets of the form $g_i \exp(V_i) \cdot z_0$ where each $V_i \subset \mathfrak{g}$ belongs to a fixed slice for the exponential map $\mathfrak{g} \rightarrow G/H$, in a neighborhood of the origin, and where $g_i \exp(V_i) \subset U$. Then UD contains the disjoint union of the sets $g_i \exp(V_i)D$, and for each of these we find

$$\text{vol}_Z(g_i \exp(V_i) \cdot z_0) \text{vol}_H(D) = \text{vol}_G(g_i \exp(V_i)D). \quad \square$$

Proof of Proposition 4.3. We first assume that $x = \mathbf{1}$. The lower bound is already established so that we can focus on the upper bound.

Let $\Xi \subset \mathfrak{h}$ be a compact symmetric neighborhood of $0 \in \mathfrak{h}$ such that $\Xi \rightarrow H, X \mapsto \exp(X)$ is diffeomorphic onto its image. It follows from Lemma 4.4 that there is a constant $C_0 > 0$ such that

$$\text{vol}_Z(Ba \cdot z_0) \leq C_0 \text{vol}_G(Ba \exp(\Xi))$$

Let B_A be a ball in A_Z . In view of (2.9), (2.7), and Corollary 3.3, it is then sufficient to show that $\text{vol}_G(KB_A a \exp(\Xi)) \leq C a^{-2\rho_u}$ for all $a \in A_Z^-$. Now

$$\text{vol}_G(KB_A a \exp(\Xi)) = \text{vol}_G \Phi_a(K \times B_A \times \Xi)$$

where

$$\Phi_a : K \times B_A \times \Xi \rightarrow G, \quad (k, b, X) \mapsto kb \exp(\text{Ad}(a)X).$$

We wish to find a uniform bound for the differential of Φ_a at all $(k, b, X) \in K \times B_A \times \Xi$ and for all $a \in A_Z^-$. For that we recall the isomorphism (2.12). Let $X = W + Y + T(Y) \in \mathfrak{h}$ where $W \in \mathfrak{l} \cap \mathfrak{h}$, $Y \in \bar{\mathfrak{u}}$, then

$$(4.3) \quad \text{Ad}(a)X = W + \text{Ad}(a)(Y + T(Y)).$$

Observe further that $\mathfrak{k} + \mathfrak{a} + \mathfrak{h}_{\lim} = \mathfrak{g}$. Let W_1, \dots, W_k be a basis of $\mathfrak{l} \cap \mathfrak{h}$, Y_1, \dots, Y_m be a basis of $\bar{\mathfrak{u}}$, V_1, \dots, V_r a basis of \mathfrak{a}_Z . Finally let U_1, \dots, U_s be independent elements from \mathfrak{k} , such that

$$W_1, \dots, W_k, Y_1, \dots, Y_m, V_1, \dots, V_r, U_1, \dots, U_s$$

is a basis of \mathfrak{g} . To simplify notation we write $\mathbf{U} = U_1 \wedge \dots \wedge U_s$ etc. For $\mathbf{Y} \in \bigwedge^k \bar{\mathfrak{u}}$ we use the notation $\mathbf{Y} + T(\mathbf{Y})$ for

$$(I + T)(\mathbf{Y}) = (\mathbf{Y}_1 + \mathbf{T}(\mathbf{Y}_1)) \wedge \dots \wedge (\mathbf{Y}_m + \mathbf{T}(\mathbf{Y}_m)) \in \bigwedge^k \mathfrak{h}.$$

Then

$$(4.4) \quad \text{Ad}(a)(\mathbf{Y} + T(\mathbf{Y})) = a^{-2\rho_{\mathfrak{u}}}(\mathbf{Y} + R_Y(a))$$

with $R_Y(a)$ bounded on A_Z^- .

We obtain a volume form ω on \mathfrak{g} by

$$\omega = \mathbf{W} \wedge \mathbf{Y} \wedge \mathbf{V} \wedge \mathbf{U}.$$

Note that by (4.3) the determinant of $d\Phi_a(\mathbf{1}, \mathbf{1}, 0)$ is given by

$$(\det d\Phi_a(\mathbf{1}, \mathbf{1}, 0)) \cdot \omega = \mathbf{W} \wedge [\text{Ad}(a)(\mathbf{Y} + T(\mathbf{Y}))] \wedge \mathbf{V} \wedge \mathbf{U}.$$

Hence by (4.4) the Jacobian of Φ_a at $(\mathbf{1}, \mathbf{1}, 0)$ is bounded by $C_2 a^{-2\rho_{\mathfrak{u}}}$ with $C_2 > 0$ a constant which is independent of $a \in A_Z^-$. As the formulas show, the bound is not changed by small distortions: for $\det d\Phi_a(k, b, \xi)$ with $k \in K$, $b \in B_A$ and $\xi \in \Xi$ the same bound holds true. This proves the proposition for $x = \mathbf{1}$.

Recall that elements $x \in F$ decompose as $x = x_1 x_2$ with $x_2 \in N_G(H)$ and $x_1 \in \exp(i\mathfrak{a}_Z)H_{\mathbb{C}} \cap G$. A quick inspection of the proof above yields the same bounds with arbitrary $x \in F$. \square

5. HARISH-CHANDRA SCHWARTZ SPACE ON Z

Let $\Gamma_Z < A_Z$ be a lattice, that is $\log \Gamma_Z \subset \mathfrak{a}_Z$ is a lattice in the vector space \mathfrak{a}_Z . Set $\Gamma_Z^- := \Gamma_Z \cap A_Z^-$. After enlarging B if necessary we can assume that

$$Z = B\Gamma_Z^- F \cdot z_0.$$

With this we have the domination

$$(5.1) \quad \int_Z f(z) dz \leq \sum_{\gamma \in \Gamma_Z^-, x \in F} \int_{B\gamma x \cdot z_0} f(z) dz$$

for every non-negative measurable function on Z . Note also that

$$(5.2) \quad \sum_{\gamma \in \Gamma_Z^-} (1 + \|\log \gamma\|)^{-s} < \infty$$

for $s > \dim \mathfrak{a}_Z$.

For $u \in \mathcal{U}(\mathfrak{g})$ and a smooth function f on Z we write $L_u f$ for the left derivative of f with respect to u . With consider two families of semi-norms on $C_c^\infty(Z)$,

$$p_{n,u}(f) := \|(1 + \mathbf{w})^n L_u f\|_2 \quad (u \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N})$$

and

$$q_{n,u}(f) := \sup_{z \in Z} (1 + \mathbf{w}(z))^n \sqrt{\mathbf{v}(z)} |L_u f(z)| \quad (u \in \mathcal{U}(\mathfrak{g}), n \in \mathbb{N}).$$

Proposition 5.1. *The two families $(p_{n,u})$ and $(q_{n,u})$ define the same locally convex topology on $C_c^\infty(Z)$.*

Proof. We apply (5.1) to the definition of $p_{n,u}(f)$. It follows from Lemma 4.1 that $\int_{By \cdot z_0} \mathbf{v}(z)^{-1} dz$ is a bounded function of $y \in Z$. Using (5.2) we then obtain $p_{n,u}(f) \leq C q_{m,u}(f)$ when $s = 2m - 2n > \dim \mathfrak{a}_Z$.

The invariant Sobolev lemma (see [2], “key lemma” in section 3.4), provides a domination in the opposite direction. \square

The completion of $C_c^\infty(Z)$ with respect to either family is denoted $\mathcal{C}(Z)$ and called the *Harish-Chandra Schwartz space* of Z . The space $\mathcal{C}(Z)$ was introduced by Harish-Chandra for $Z = G = G \times G/G$ (see [21], Sect. 7.1, for a simplified exposition), and it was extended to symmetric spaces by van den Ban in [1], Sect. 17.

By the local Sobolev lemma we see that $\mathcal{C}(Z) \subset C^\infty(Z)$.

Proposition 5.2. *The inclusion*

$$\mathcal{C}(Z) \hookrightarrow L^2(Z)$$

is fine, i.e. there exists a continuous Hermitian norm p on $\mathcal{C}(Z)$ such that the completion $\mathcal{C}(Z)_p$ of $(\mathcal{C}(Z), p)$ gives rise to a Hilbert-Schmidt embedding $\mathcal{C}(Z)_p \rightarrow L^2(Z)$. Moreover $\mathcal{C}(Z)$ is nuclear.

Proof. It follows from (5.2) that the weight

$$z \mapsto (1 + \mathbf{w}(z))^n$$

is summable in the sense of [2], Sect. 3.2, provided that $n > \dim \mathfrak{a}_Z$. Hence [2], Theorem 3.2, applies to $\mathcal{C}(Z)$ and the assertions follow. \square

5.1. Z -tempered Harish-Chandra modules. Let V be a Harish-Chandra module for (\mathfrak{g}, K) and V^∞ its unique smooth moderate growth Fréchet globalization, see [3]. We denote by $V^{-\infty}$ the strong dual of V^∞ , and by $(V^{-\infty})^H$ its subspace of H -fixed vectors which we recall is finite dimensional (see [15] or [18]).

Attached to $\eta \in (V^{-\infty})^H$ and $v \in V^\infty$ is the matrix coefficient

$$m_{v,\eta}(gH) := \eta(\pi(g^{-1})v) \quad (g \in G)$$

which is a smooth function on Z .

Definition 5.3. Let $Z = G/H$ be a unimodular real spherical space, V a Harish-Chandra module for (\mathfrak{g}, K) and $\eta \in (V^{-\infty})^H$. The pair (V, η) is called Z -tempered provided that there exists an $n \in \mathbb{Z}$ such that for all $v \in V$ one has

$$(5.3) \quad \sup_{z \in Z} |m_{v, \eta}(z)| \sqrt{\mathbf{v}(z)} (1 + \mathbf{w}(z))^n < \infty.$$

Remark 5.4. Note that by Proposition 3.4(2) and Proposition 4.3 the bound (5.3) can equivalently be expressed as follows.

For all $v \in V$ there exists a constant $C_v > 0$ such that

$$(5.4) \quad |m_{v, \eta}(\omega a x \cdot z_0)| \leq C_v a^{\rho_Q} (1 + \|\log a\|)^n$$

for all $\omega \in \Omega$, $x \in F$ and $a \in A_Z^-$.

We recall the abstract Plancherel theorem for $L^2(Z)$:

$$(5.5) \quad L^2(Z) \simeq \int_{\widehat{G}} \mathcal{M}_\pi \otimes \mathcal{H}_\pi \, d\mu(\pi).$$

Here $\mathcal{M}_\pi \subset (V_\pi^{-\infty})^H$ is a subspace, the multiplicity space. The measure class of μ is unique. Specific Plancherel measures result from choices of the inner product on the finite dimensional multiplicity spaces \mathcal{M}_π .

The isomorphism (5.5) is given by the abstract inverse-Fourier-transform which assigns to a smooth section $\widehat{G} \ni \pi \mapsto \eta_\pi \otimes v_\pi \in \mathcal{M}_\pi \otimes \mathcal{H}_\pi^\infty$ the (generalized) function on Z

$$z \mapsto \int_{\widehat{G}} m_{v_\pi, \eta_\pi}(z) \, d\mu(\pi) \quad (z \in Z).$$

Let us denote by V_π the Harish-Chandra module of the unitary representation (π, \mathcal{H}_π) .

Proposition 5.1 combined with [2] (see [6] for a nice summary of the results in [2]) then yields the following characterization of the spectrum of $L^2(Z)$.

Proposition 5.5. Let Z be a unimodular real spherical space with Plancherel measure μ . For μ -almost every $\pi \in \widehat{G}$ and each $\eta_\pi \in \mathcal{M}_\pi$ the pair (V_π, η_π) is Z -tempered.

Remark 5.6. The multiplicity space \mathcal{M}_π can be a proper subspace of $(V_\pi^{-\infty})^H$. This happens for example for the Lorentzian symmetric space $G/H = \mathrm{SO}_0(n, 1)/\mathrm{SO}_0(n-1, 1)$ when $n \geq 4$, in which case it can be shown with methods from [20] that there exists an irreducible Harish-Chandra module V which embeds into $L^2(G/H)$ with multiplicity one, but with $\dim(V_\pi^{-\infty})^H = 2$.

6. BOUNDS FOR GENERALIZED MATRIX COEFFICIENTS

In this section we assume that H is a closed subgroup of G with algebraic Lie algebra and with finitely many components.

Let V be a Harish-Chandra module for (\mathfrak{g}, K) . We say that a norm q on V^∞ is G -continuous, provided that the completion of (V^∞, q) is a Banach representation of G . For a G -continuous norm on V^∞ we denote by q^* its dual norm. Note that for all $\eta \in V^{-\infty}$ there exists a G -continuous norm p such that $p^*(\eta) < \infty$.

Remark 6.1. Note that for any compact subset $\Omega \subset G$ and $v \in V^\infty$ we have $\sup_{g \in \Omega} q(\pi(g)v) < \infty$ for all G -continuous norms on V^∞ .

In [21] 4.3.5 or [16] equation (3.5) one associates to V an exponent $\Lambda_V \in \mathfrak{a}^*$, the definition of which we recall and adapt to the current set-up. Consider the finite-dimensional A -module $V/\bar{\mathfrak{n}}V$ and its spectrum of weights, say $\mu_1, \dots, \mu_k \in \mathfrak{a}_\mathbb{C}^*$. Let $H_1, \dots, H_n \in \mathfrak{a}$ be the basis elements which are dual to the simple roots for \mathfrak{n} . Then

$$\Lambda_V(H_i) := \max_{1 \leq j \leq k} \operatorname{Re} \mu_j(H_i) \quad (1 \leq i \leq n).$$

Further we attach the integer $d_V \in \mathbb{N}_0$ as in [16].

Remark 6.2. Later we shall use the following property of the exponent Λ_V , which is a consequence of the Howe-Moore theorem (see [19], p. 447). If \mathfrak{g} is simple and π is non-trivial unitary, then

$$\Lambda_V \in \operatorname{Int}(\mathfrak{a}^+)^*,$$

the interior of the dual cone of \mathfrak{a}^+ .

Theorem 6.3. *Let $Z = G/H$ be as above, and suppose for the minimal parabolic subgroup $P \subset G$ that PH is open in G . Let V be a Harish-Chandra module. Fix a G -continuous norm p on V^∞ . Then there exists a G -continuous norm q on V^∞ such that*

$$(6.1) \quad |m_{v,\eta}(a \cdot z_0)| \leq q(v)p^*(\eta)a^{\Lambda_V}(1 + \|\log a\|)^{d_V}$$

for $a \in A^-$, $v \in V^\infty$ and $\eta \in (V^{-\infty})^H$.

Remark 6.4. Note that by applying the theorem to the subgroup $H^x = xHx^{-1}$ and the H^x -fixed distribution vector $\eta^x = x.\eta$, one obtains for each $x \in G$ such that PxH is open a similar domination of $m_{v,\eta}(ax \cdot z_0)$ with a norm q that depends on x .

Remark 6.5. In (6.1) the exponent Λ_V can be replaced by one with a more refined definition that relates to Z . Recall from Section 2 the parabolic subgroup $Q = LU \supset P$. If in the definition of Λ_V one replaces the weights of $V/\bar{\mathfrak{n}}V$ by those of the smaller space $V/(\mathfrak{l} \cap \mathfrak{h} + \bar{\mathfrak{u}})V$, one

obtains an ‘ H -spherical’ exponent $\Lambda_{H,V} \in \mathfrak{a}^*$. With the technique of [18] Thm. 3.2 the bound can then be improved with $\Lambda_{H,V}$ in place of Λ_V . As this is not currently needed the details are omitted.

Proof. The proof is a development of the proof of [16], Th. 3.2. In order to compare with [16], it is convenient to rewrite the statement above. We shall assume

$$(6.2) \quad \bar{P}H \text{ open}$$

and instead of (6.1) prove for all $a \in A^+$ and v, η as before that

$$(6.3) \quad |m_{v,\eta}(a \cdot z_0)| \leq q(v)p^*(\eta)a^{\Lambda_V}(1 + \|\log a\|)^{d_V}.$$

In [16] it is assumed that PH is open and that H is reductive. However, the first step of the given proof consists of the observation that then $\bar{P}H$ is also open. With (6.2) this step is superfluous. As the assumption that H is reductive is only used for that step, it thus follows from the theorem in [16] that for each $v \in V$ there exists a constant $C > 0$ such that

$$(6.4) \quad |m_{v,\eta}(a \cdot z_0)| \leq Cp^*(\eta)a^{\Lambda_V}(1 + \|\log a\|)^{d_V}$$

for all $a \in A^+$.

The contents of the extended version (6.3) is that we can replace C in (6.4) by a G -continuous norm.

For that we just need to add an extra ingredient to the proof in [16]. The new ingredient is the Casselman comparison theorem (see Remark 6.6 below). Let $\Pi \subset \Sigma^+$ the set of simple roots. For a subset $F \subset \Pi$ one associates a standard parabolic subalgebra $\mathfrak{p}_F := \mathfrak{m}_F + \mathfrak{a}_F + \mathfrak{n}_F$ with $\mathfrak{a}_F = \{X \in \mathfrak{a} \mid (\forall \alpha \in F)\alpha(X) = 0\}$ etc. The comparison theorem asserts in particular that $\mathfrak{n}_F V^\infty$ is closed in V^∞ . Let X_1, \dots, X_n be a basis of \mathfrak{n}_F and consider the surjective map of Fréchet spaces

$$\mathcal{T} : \mathfrak{n}_F \otimes V^\infty \rightarrow \mathfrak{n}_F V^\infty, \quad \sum_{j=1}^n X_j \otimes v_j \mapsto \sum_{j=1}^n X_j \cdot v_j.$$

The open mapping theorem implies that for every neighborhood U of 0 in $\mathfrak{n}_F \otimes V^\infty$ there exists a neighborhood \tilde{U} of 0 in $\mathfrak{n}_F V^\infty$ such that $\tilde{U} \subset \mathcal{T}(U)$. Hence for every G -continuous norm q on V^∞ there exists a G -continuous norm \tilde{q} on V^∞ such that for all $w \in \mathfrak{n}V^\infty$ there exist $v_1, \dots, v_n \in V^\infty$ with $w = \mathcal{T}(\sum_{j=1}^n X_j \otimes v_j)$ and $q(v_j) \leq \tilde{q}(w)$. Having said that the proof is a simple modification of [16]: one obtains a quantitative version of the key-step leading to (3.11) in the proof of Theorem 3.2 in [16]. \square

In view of Remarks 6.1 and 6.4, the polar decomposition (2.9) would allow us to obtain a global bound from the upper bound (6.1), if it were on A_Z^- and not on the potentially smaller set A^-A_H/A_H . We recall that spherical spaces with the property

$$A_Z^- = A^-A_H/A_H$$

are called *wavefront* (see [12], Sect. 6). All symmetric spaces are wavefront.

Remark 6.6. For a Harish-Chandra module V for (\mathfrak{g}, K) and $F \subset \Pi$ we obtain an induced Harish-Chandra module $V/\mathfrak{n}_F V = H_0(V, \mathfrak{n}_F V)$ for the pair (\mathfrak{g}_F, K_F) where $\mathfrak{g}_F = \mathfrak{m}_F + \mathfrak{a}_F$. Likewise all higher homology groups $H_p(V, \mathfrak{n}_F)$ are Harish-Chandra modules for (\mathfrak{g}_F, K_F) .

The Casselman comparison theorem states that the \mathfrak{n}_F -homology groups $H_p(V^\infty, \mathfrak{n}_F)$ are separated (Hausdorff) and that the natural mappings $H_p(V, \mathfrak{n}_F) \rightarrow H_p(V^\infty, \mathfrak{n}_F)$ induce isomorphisms $H_p(V, \mathfrak{n}_F)^\infty \simeq H_p(V^\infty, \mathfrak{n}_F)$ for all $p \geq 0$. In particular, for $p = 0$, we obtain that $\mathfrak{n}_F V^\infty$ is closed in V^∞ . Up to present, this theorem remains unpublished, although it was applied quite often (see for instance [5], Th. 1.5). Notice that the closedness of $\mathfrak{n}_F V^\infty$ was crucial in the above proof.

An analytic version of the comparison theorem has been established (see [4], Thm. 1). It implies that all $H_p(V^\omega, \mathfrak{n}_F)$ are separated and

$$H_p(V, \mathfrak{n}_F)^\omega \simeq H_p(V^\omega, \mathfrak{n}_F) \quad (p \geq 0).$$

Here V^ω is the space of analytic vectors of V . In fact for the purposes of this paper the analytic comparison theorem is sufficient. Eventually it leads to a slight reformulation of the bounds in Theorem 6.3 in terms of analytic norms. We now describe the details.

To begin with we briefly recall the nature of the topological vector space V^ω , see [7]. We fix a G -continuous norm p on V and let V_p be the Banach completion of (V, p) . Let U be a bounded open neighborhood of 0 in $\mathfrak{g}_\mathbb{C}$ such that $\exp|_U$ is diffeomorphic onto its image in $G_\mathbb{C}$. For $\epsilon > 0$ we set $\mathcal{U}_\epsilon := \exp(U_\epsilon)$ and let $\overline{\mathcal{U}}_\epsilon$ be its closure. We denote by $V_p^\epsilon \subset V_p$ the subspace of those vectors $v \in V_p$ for which the orbit map $G \rightarrow V_p$, $g \mapsto g \cdot v$ extends to a continuous map from $G\overline{\mathcal{U}}_\epsilon$ to E which is holomorphic when restricted on \mathcal{U}_ϵ . Then V_p^ϵ becomes a Banach space with norm $p_\epsilon(v) = \max_{g \in \mathcal{U}_\epsilon} p(g \cdot v)$. Note that there are natural continuous inclusions $V_p^\epsilon \rightarrow V_p^{\epsilon'}$ for $\epsilon' < \epsilon$. Then $V^\omega = \lim_{\epsilon \rightarrow 0} V_p^\epsilon$ is the inductive limit of Banach spaces V_p^ϵ . As a topological vector space V^ω is of type DNF (dual nuclear Fréchet).

Our concern is now the continuous surjective map

$$\psi : \mathfrak{n}_F \otimes V^\omega \rightarrow \mathfrak{n}_F V^\omega.$$

As closed subspaces of DNF-spaces are DNF we infer from the open mapping theorem (which holds true for DNF-spaces, see [9], Appendix A.6) that ψ is an open mapping. Let $\mathcal{K} := \ker \psi$ and

$$\phi : \mathfrak{n}_F V^\omega \rightarrow (\mathfrak{n}_F \otimes V^\omega) / \mathcal{K}$$

the inverse map induced from ψ . We are interested in a quantitative description of the continuity of ϕ . Fix $\epsilon > 0$. As $V_p^\epsilon \rightarrow V^\omega$ is continuous, we get that $V_p^\epsilon \cap \mathfrak{n}_F V^\omega$ is closed in the Banach space V_p^ϵ and further a continuous map

$$\phi_\epsilon : V_p^\epsilon \cap \mathfrak{n}_F V^\omega \rightarrow (\mathfrak{n}_F \otimes V^\omega) / \mathcal{K}.$$

For $\delta > 0$ we set $\mathcal{K}_\delta := (\mathfrak{n}_F \otimes V_p^\delta) \cap \mathcal{K}$. Then \mathcal{K}_δ is a closed subspace of the Banach space $\mathfrak{n}_F \otimes V_p^\delta$ and we have continuous inclusions $u_\delta : (\mathfrak{n}_F \otimes V_p^\delta) / \mathcal{K}_\delta \rightarrow (\mathfrak{n}_F \otimes V^\omega) / \mathcal{K}$ with

$$(\mathfrak{n}_F \otimes V^\omega) / \mathcal{K} = \bigcup_{\delta > 0} \text{im } u_\delta.$$

Hence the Grothendieck factorization theorem (see [8], Ch. 4, Sect. 5, Th. 1) applies to the map ϕ_ϵ and we obtain an $\epsilon' > 0$ such that $\text{im } \phi_\epsilon \subset \text{im } u_{\epsilon'}$ and that the induced map between Banach spaces $V_p^\epsilon \cap \mathfrak{n}_F V^\omega \rightarrow (\mathfrak{n}_F \otimes V_p^\delta) / \mathcal{K}_\delta$ is continuous. In particular, For all $\epsilon > 0$ there exists an $\epsilon' > 0$ and a constant $C_\epsilon > 0$ such that for all $v \in \mathfrak{n}_F V$ there exists a presentation $v = \sum_\alpha X_{-\alpha} u_\alpha$ with

$$(6.5) \quad \sum_\alpha p_\epsilon(u_\alpha) \leq C_\epsilon p_{\epsilon'}(v).$$

This was the crucial topological ingredient to the proof of Theorem 6.3. The upshot is that we arrive at the same bound as in (6.1) but with q replaced by p_ϵ . For the current application these bounds are sufficient. The reason why we formulated matters in the smooth category is mainly that we consider smooth completions as more natural than analytic ones.

Remark 6.7. It is possible to obtain a bound on all of A_Z^- but not solely with the ODE-techniques used in this approach. We will return to that topic in [14].

7. PROPERTY (I)

We assume for the moment just that G is a real reductive group and $Z = G/H$ a unimodular homogeneous space. We introduce an integrability condition for matrix coefficients on Z . It has some similarity with Kazhdan's property (T).

We denote by \widehat{G} the unitary dual of G .

Definition 7.1. We say that $Z = G/H$ has property (I) provided for all $\pi \in \widehat{G}$ and $\eta \in (\mathcal{H}_\pi^{-\infty})^H$ the stabilizer H_η of η is such that $Z_\eta := G/H_\eta$ is unimodular and there exists $1 \leq p < \infty$ such that

$$(7.1) \quad m_{v,\eta} \in L^p(G/H_\eta),$$

for all $v \in \mathcal{H}_\pi^\infty$.

The following lemma shows that it suffices to have (7.1) for K -finite vectors v of any given type which occurs in π .

Lemma 7.2. Let (π, \mathcal{H}_π) be irreducible unitary, and let $\eta \in (\mathcal{H}_\pi^{-\infty})^H$ and $1 \leq p < \infty$. The following statements are equivalent:

- (1) $m_{v,\eta} \in L^p(Z)$ for all $v \in \mathcal{H}_\pi^\infty$.
- (2) $m_{v,\eta} \in L^p(Z)$ for all K -finite vectors in \mathcal{H}_π .
- (3) $m_{v,\eta} \in L^p(Z)$ for some K -finite vector $v \neq 0$.

Proof. Let V be the Harish-Chandra module of (π, \mathcal{H}_π) , i.e. the space of K -finite vectors. According to Harish-Chandra, V is an irreducible (\mathfrak{g}, K) -module. The map $v \mapsto m_{v,\eta}$ is equivariant $V \rightarrow C^\infty(Z)$.

We first establish “(3) \Rightarrow (2)”. Let $v \in V$ be non-zero with $m_{v,\eta} \in L^p(Z)$, then v generates V , and (2) is equivalent with the statement that $m_{v,\eta} \in L^p(Z)^\infty$.

Let E be the closed G -invariant subspace of $L^p(Z)$ generated by $m_{v,\eta}$. As the left action on $L^p(Z)$ is a Banach representation, the same holds for E . The Casimir element \mathcal{C} acts by a scalar on V , hence it acts (in the distribution sense) on E by the same scalar. It follows that all K -finite vectors in E are smooth for the Laplacian Δ associated to \mathcal{C} . Thus any K -finite vector of E belongs to $E^\infty \subset L^p(Z)^\infty$ by [3], Prop. 3.5.

Finally “(2) \Rightarrow (1)” follows from the Casselman-Wallach globalization theorem (see [3]), which implies that the map $v \mapsto m_{v,\eta}$, $V \rightarrow L^p(Z)$, extends to $\mathcal{H}_\pi^\infty \rightarrow L^p(Z)^\infty$. \square

In the definition of property (I) we have to take into account that the stabilizer H_η inflates H . To discuss this efficiently it is useful to have an appropriate notion of factorization for Z .

7.1. Factorization. If there exists a closed subgroup $H \subset H^* \subset G$ with $Z^* = G/H^*$ unimodular, then we call Z *factorizable*, and call

$$Z \mapsto Z^*, \quad gH \mapsto gH^*$$

a *factorization* of Z . We call the factorization *proper* if $\dim H < \dim H^* < \dim G$ and *co-compact* if H^*/H is compact.

Example 7.3. 1) Irreducible symmetric spaces do not have proper factorizations. In fact, if $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair and $\mathfrak{h} \subsetneq \mathfrak{h}^* \subsetneq \mathfrak{g}$ a factorization, then $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2$ and $\mathfrak{h}^* = \mathfrak{h}_1 \times \mathfrak{g}_2$ for some decomposition of \mathfrak{g} .

2) Let $G^n := G \times \cdots \times G$ denote the direct product of n copies of G . Suppose that G is simple. The homogeneous space $Z = G^n / \text{diag}(G)$ with the diagonal subgroup $\text{diag}(G) = \{(g, \dots, g) | g \in G\}$ is factorizable if and only if $n > 2$. For example, for $n = 3$, we can take $H^* = \{(g_1, g_2, g_2) | g_1, g_2 \in G\} \simeq G \times G$ and permutations thereof.

3) Let $G/H = \text{SO}(8, \mathbb{C})/G_2(\mathbb{C})$. This space is spherical but not wavefront. The symmetric space $G/H^* = \text{SO}(8, \mathbb{C})/\text{SO}(7, \mathbb{C})$ is a factorization.

Note that one has $H_\eta \supset H$ for every $\eta \in (V^{-\infty})^H$. However it is a priori not clear that G/H_η is unimodular if G/H was unimodular. Here is an example:

Example 7.4. Suppose that $Z = G/N$ with N the unipotent radical of a minimal parabolic $P = MAN$. Assume in addition that G is simple and that $\dim A \geq 2$. We let V be a generic irreducible K -spherical unitary principal series. Then $\dim(V^{-\infty})^N = \#\mathcal{W}$ with \mathcal{W} the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. As V was supposed to be generic, the action of A on $(V^{-\infty})^N$ is semi-simple. Let $\eta \in (V^{-\infty})^N$ such that $a \cdot \eta = \chi(a)\eta$ for $a \in A$ and a character $\chi : A \rightarrow \mathbb{C}^*$. If we denote by $A_\chi := \ker \chi$, then $H_\eta = A_\chi N$. As G is simple and $\dim A \geq 2$ it follows that A_χ acts non-trivially on N . We may assume that χ is generic enough (not a multiple of ρ) so that A_χ acts in a non-unimodular fashion on N . Thus Z_η is not unimodular.

In case Z is a wavefront real spherical space with H reductive, then the factorizations which are of type G/H_η will turn out to be of a special simple shape. Let

$$(7.2) \quad \mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$$

be a decomposition into simple ideals and one dimensional ideals. Here the simple ideals \mathfrak{g}_i are unique (up to the order of terms). For a subset $I \subset \{1, \dots, k\}$ we set

$$\mathfrak{h}_I := \mathfrak{h} + \bigoplus_{j \in I} \mathfrak{g}_j.$$

Further we set $H_I = H \prod_{j \in I} G_j$ with $G_j \triangleleft G$ the connected normal subgroup of G with Lie algebra \mathfrak{g}_j . If H is reductive and $H^* < G$ is a subgroup with Lie algebra $\mathfrak{h}^* = \mathfrak{h}_I$ for some $I \subset \{1, \dots, k\}$ and some decomposition (7.2), then we call $Z^* := G/H^*$ a *basic factorization*. Note that H reductive implies H^* reductive in this case.

Example 7.5. Suppose that $Z = G \times G \times G / \text{diag}(G)$ with G simple. Irreducible unitary representations of $G \times G \times G$ are tensor products $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$. If π is non-trivial and H -spherical and if one constituent, say π_1 , is trivial, then $H_\eta = \{(g_1, g_2, g_2) \mid g_1, g_2 \in G\}$ is basic.

In the sequel we need a weaker notion of basic factorization. We call a factorization G/H^\star of G/H *weakly basic* if it is built by a sequence of factorizations each which is either co-compact or basic.

7.2. Main theorem on property (I). In this last section we resume the assumption of Section 6, that is, H is a closed subgroup of G with algebraic Lie algebra and with finitely many components. We call $Z = G/H$ real spherical when PH is open for some minimal parabolic subgroup. If this is the case, then we say that G/H is wavefront if its factorization with the Zariski closure of H is wavefront.

We can now state one of the main results of the paper.

Theorem 7.6. *Let $Z = G/H$ be a wavefront real spherical space with H reductive. Then Z has property (I). Moreover, $Z_\eta = G/H_\eta$ is a weakly basic factorization of Z , for every unitarizable Harish-Chandra module V and $\eta \in (V^{-\infty})^H$.*

Proof. Note that any basic factorization Z^\star of Z will satisfy the same assumptions as requested for Z . Proceeding by induction, we may thus assume that all proper basic factorizations Z^\star have property (I). We may also assume that \mathfrak{h} contains no non-trivial ideal of \mathfrak{g} and that \mathfrak{g} is semi-simple.

Let $\pi \in \widehat{G}$ be non-trivial and let $f = m_{v,\eta}$ with non-zero vectors $\eta \in (\mathcal{H}_\pi^{-\infty})^H$ and $v \in \mathcal{H}_\pi^\infty$. In view of Lemma 7.2 it is sufficient for property (I) that G/H_η is unimodular and that $f \in L^p(G/H_\eta)$ for some $1 \leq p < \infty$.

As Z is wavefront we have $Z = \Omega A^- F \cdot z_0$. Note that we may assume that $\Omega \subset B$ where $B \subset G$ is a ball. Furthermore, we may assume that G is semisimple.

Assume first that π is non-trivial on every non-trivial connected normal subgroup of G . Then by Remark 6.2 the exponent Λ_V of π is contained in $\text{Int}(\mathfrak{a}^+)^*$, i.e. if $\|\cdot\|$ is a norm on \mathfrak{a} , then there exists a constant $C > 0$ such that

$$-\Lambda_V(X) \geq C\|X\| \quad (X \in \mathfrak{a}^-).$$

We apply (5.1) to the integral of $|f|^p$. For each $\gamma \in \Gamma_Z^-$ and $x \in F$ we find from Theorem 6.3 and Remark 6.4

$$|f(g\gamma x \cdot z_0)| \leq q(g^{-1} \cdot v) p^*(x \cdot \eta) \gamma^\Lambda (1 + \|\log(\gamma)\|)^d$$

and hence by (4.2) and Remark 6.1

$$\int_{B_{\gamma x \cdot z_0}} |f(z)|^p dz \leq C \gamma^{p\Lambda_V - 2\rho_u} (1 + \|\log(\gamma)\|)^{pd}$$

for some constant $C > 0$. As $\Lambda_V \in \text{Int}(\mathfrak{a}^+)^*$, this can be summed over Γ_Z^- for p sufficiently large, and then $f \in L^p(Z)$.

We claim that H_η/H is compact. Otherwise we find a sequence $\omega_n a_n x \cdot z_0 \in H_\eta/H$ with $\omega_n \in \Omega$ and $a_n \rightarrow \infty$ in A_Z^- . But this contradicts the bound from Theorem 6.3. Hence Z_η is a co-compact factorization. In particular, it is then unimodular and weakly basic. This establishes the properties requested in Definition 7.1 for this case.

On the other hand, assume G is not simple and π is 1 on some non-trivial normal subgroup $S \triangleleft G$. This subgroup is contained in H_η but not in H , hence $Z \rightarrow G/SH$ is a proper basic factorization. It follows from our inductive hypothesis that G/SH has property (I) and that $G/(SH)_\eta$ is weakly basic. Since $(SH)_\eta = H_\eta$ we obtain the assertions in the theorem also for this case. \square

This result also has a geometric converse.

Proposition 7.7. *Let Z be a wavefront real spherical space with H reductive. Then every factorization is weakly basic.*

Proof. Let Z^* be a factorization of Z . Theorem 7.6 implies that it is sufficient to exhibit an irreducible unitary representation (π, \mathcal{H}) of G with $\eta \in (\mathcal{H}^{-\infty})^{H^*} \subset (\mathcal{H}^{-\infty})^H$ such that H_η/H^* is compact. As Z^* is unimodular, the left regular representation of G on the Hilbert space $L^2(Z^*)$ is unitary. Every generic irreducible unitary representation (π, \mathcal{H}) which is weakly contained in $L^2(Z^*)$ has the requested property. \square

Remark 7.8. The geometric assumption that H is reductive is in some sense natural in the context of harmonic analysis. In [11] it is shown that if $Z = G/H$ is unimodular and H is self-normalizing, then H is reductive.

REFERENCES

- [1] E. van den Ban, *The principal series for a reductive symmetric space II. Eisenstein integrals*, J. Funct. Anal. **109** (1992), 331–441.
- [2] J. Bernstein, *On the support of Plancherel measure*, Jour. of Geom. and Physics **5**, No. 4 (1988), 663–710.
- [3] J. Bernstein and B. Krötz, *Smooth Fréchet globalizations of Harish-Chandra modules*, Israel J. Math. **199** (2014), 45–111.
- [4] T. Bratten, *A comparison theorem for Lie algebra homology groups*, Pacific journal of mathematics **182** (1), 1998, 23–36.

- [5] U. Bunke and M. Olbrich, *Cohomological properties of the canonical globalizations of Harish-Chandra modules*, Annals of Global Analysis and Geometry 09/1997, **15**(5), 401–418.
- [6] J. Carmona and P. Delorme, *Base méromorphe de vecteurs distributions H -invariants pour les séries principales généralisées d'espaces symétriques réductifs: equation fonctionnelle*, J. Funct. Anal. **122** (1) (1994), 152–221.
- [7] H. Gimperlein, B. Krötz, and H. Schlichtkrull, *Analytic representation theory of Lie groups: General theory and analytic globalizations of Harish-Chandra modules*, Compos. Math. **147** (2011), 1581–1607.
- [8] A. Grothendieck, "Topological vector spaces", Gordon and Breach, 1973.
- [9] H. Hecht and J. Taylor, *Analytic localization of group representations*, Advances in Math., **79**, (1990), 139–212.
- [10] H. Hecht and J. Taylor, *A remark on Casselman's comparison theorem*, Geometry and representation theory of real and p -adic groups (Córdoba, 1995), 139–146, Progr. Math., **158**, Birkhäuser Boston, Boston, MA, 1998.
- [11] F. Knop and B. Krötz, *Reductive group actions*, in preparation
- [12] F. Knop, B. Krötz, E. Sayag and H. Schlichtkrull, *Simple compactifications and polar decomposition for real spherical spaces*, Selecta Math. (N.S.) **21** (2015), 1071–1097.
- [13] F. Knop, B. Krötz and H. Schlichtkrull, *The local structure theorem for real spherical varieties*, Compositio Math. **151** (2015), 2145–2159.
- [14] ———, *The tempered spectrum of a real spherical space*, arXiv 1509.03429.
- [15] T. Kobayashi and T. Oshima, *Finite multiplicity theorems for induction and restriction*, Advances in Math. **248** (2013), 921–944.
- [16] B. Krötz, E. Sayag and H. Schlichtkrull, *Decay of matrix coefficients on reductive homogeneous space of spherical type*, Math. Z., **278** (2014), 229–249.
- [17] ———, *The harmonic analysis of lattice counting on real spherical spaces*, arXiv 1409.0258.
- [18] B. Krötz and H. Schlichtkrull, *Multiplicity bounds and the subrepresentation theorem for real spherical spaces*, arXiv:1309.0930, Trans. AMS, <http://dx.doi.org/10.1090/S0002-9947-2014-06427-1>.
- [19] Z. Rudnick and H. Schlichtkrull, *Decay of eigenfunctions on semisimple symmetric spaces*, Duke Math. J. **64** (1991), 445–450.
- [20] H. Schlichtkrull, *Eigenspaces of the Laplacian on hyperbolic spaces: Composition series and integral transforms*, J. Funct. Anal. **70** (1987), 194–219.
- [21] N. Wallach, "Real Reductive Groups I", Academic Press, 1988.